Generalized matrix Ansatz in the multispecies exclusion process – partially asymmetric case

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Abstract. We investigate one of the simplest multispecies generalization of the asymmetric simple exclusion process on a ring. This process has a rich combinatorial spectral structure and a matrix product form for the stationary state. In the totally asymmetric case operators that conjugate the dynamics of systems with different numbers of species were obtained by the authors and reported recently [4]. The existence of such nontrivial operators was reformulated as a representation problem for a specific quadratic algebra (generalized matrix Ansatz). In the present work, we construct the family of representations explicitly for the partially asymmetric case. This solution cannot be obtained by a simple deformation of the totally asymmetric case.

1. Introduction

The fundamental quest of statistical mechanics is to derive macroscopic behavior from microscopic laws. In this respect, the asymmetric simple exclusion process (ASEP) has been playing a central role for the last two decades [12, 35]. The ASEP is one of the few models for which the hydrodynamic limit has been proved mathematically, using elaborate large deviation techniques [20, 34]. In nonequilibrium statistical mechanics, the exclusion process, being one of the simplest non-trivial interacting particle process that one can imagine, has reached the status of a paradigm [19, 33]. The ASEP is a lattice-gas where each particle is a (possibly biased) random walker hopping from a site to one of the neighboring locations only if the target site is empty. This exclusion constraint mimics a hard-core interaction amongst particles. If the hopping rates are not isotropic, a non-vanishing flow of particles is transported through the system which is therefore permanently driven out of equilibrium. Despite its simplicity, the ASEP displays a deep mathematical structure that has enabled many researchers to perform precise analytical studies. These results can be used as benchmarks in the ongoing construction of a general theory of nonequilibrium systems [7, 10, 11, 12, 16, 23, 21, 28, 31].

The standard exclusion process involves particles and holes hopping on a one-dimensional lattice, although many variants have been studied [7, 31]. A straightforward generalization of the ASEP is to consider the case of multiple species of particles with hierarchical dynamical rules [1, 5, 2, 24, 36]. In fact, such a model with N different species of particles automatically appears when one couples N standard exclusion processes [18] in a natural way. The N-ASEP provides a fundamental example of a multicomponent non-equilibrium process; it has highly non-trivial steady states, which are not Gibbs states in general and depend on boundary conditions [3, 7, 24, 32].

For the totally asymmetric case (N-TASEP), the stationary state was constructed combinatorially by Ferrari and Martin [15]. This construction was restated as a matrix product Ansatz in [14] and was generalized in [24] to the partially asymmetric case (N-PASEP).

In a recent study [4] of the N-TASEP, the matrix product Ansatz was generalized in order to construct a conjugation operator that embeds the $(N-1)$-TASEP in the N-TASEP. By considering the whole family of N-TASEP processes, with varying N, a network of mappings can be constructed (corresponding to an underlying partially ordered set –*poset*– structure). It was shown that the Ferrari and Martin construction was a special case of a more general algorithm, corresponding to a generalized matrix Ansatz that allows one to lift information from a system containing less species of particles to a system containing more species, by recursively splitting identical classes of particles into different species. Moreover, the information that is obtained is not restricted to the steady states but also affects subsets of the spectrum. However, the results of [4] were only valid for the N-TASEP: the general Ferrari and Martin algorithm cannot directly be applied to the N-ASEP model. Besides, the representations of the generalized matrix Ansatz that were obtained for the N-TASEP could not be deformed to the N-ASEP in general. The purpose of the present work is to fill this gap: we shall explain how to construct all the conjugation operators by providing an explicit representation for the quadratic algebras involved in the N-ASEP poset structure. We shall also explain how the N-ASEP relates to the Perk-Schultz model and investigate the relationship between the conjugation operators and the

Perk-Schultz transfer matrix as well.

The outline of the present work is as follows. In Section 2, we define the model, discuss its basic features and briefly review the spectral inclusion properties and the generalized matrix Ansatz. The Ansatz starts with a local relation which gives a quadratic algebra. The representation for this algebra in the totally asymmetric case [4], however, cannot be extended directly to the partially asymmetric case (the N-PASEP). We will present a family of representations for the PASEP in Section 3. In Section 4, we explain the relevance of our results to the Perk-Schultz model. We present the conclusion of the present work in Section 5.

2. Known results about the N-ASEP model

2.1. Definition of the model

Consider a ring $\mathbb{Z}_L$ with L sites, where a variable (local state) $k_i \in \{1, \ldots, N+1\}$ is assigned to each site $i \in \mathbb{Z}_L$. Nearest neighbor pairs of local states JK are interchanged $JK \to KJ$ with the transition rate

$$\Theta(J - K) = \begin{cases} p & \text{if } J < K, \\ q & \text{if } J > K \end{cases} \tag{1}$$

with $\Theta(0) = 0$. Without loss of generality, we set $0 \leq q \leq p = 1$. We say that the site i is occupied by a Jth class particle if $k_i = J \leq N$. We regard the site i as being empty if $k_i = N + 1$. When $q = 0$, the model is totally asymmetric and is called the N-TASEP; for $0 < q < 1$, the model is partially asymmetric and is called the N-PASEP. The special case $q = 1$, corresponds to the symmetric simple exclusion process (SSEP).

The dynamics of the N-ASEP can be encoded in a master equation. Let $\{|1\rangle, \ldots, |N+1\rangle\}$ be the basis of the single-site space $\mathbb{C}^{N+1}$ and $|k_1 \ldots k_L\rangle$ be the tensor product $|k_1\rangle \otimes \cdots \otimes |k_L\rangle \in (\mathbb{C}^{N+1})^{\otimes L}$. In terms of the probability vector

$$|P(t)\rangle = \sum_{1 \leq k_i \leq N+1} P(k_1 \cdots k_L; t)|k_1 \cdots k_L\rangle, \tag{2}$$

the master equation that governs the system is expressed as

$$\frac{d}{dt}|P(t)\rangle = M^{(N)}|P(t)\rangle. \tag{3}$$

The linear operator $M^{(N)}$ has the form

$$M^{(N)} = \sum_{i \in \mathbb{Z}_L} \left(M_{\text{Loc}}^{(N)}\right)_{i,i+1} \tag{4}$$

where the local operators $\left(M_{\text{Loc}}^{(N)}\right)_{i,i+1}$ act on the ith and the $(i+1)$th components of the tensor product and are given by

$$M_{\text{Loc}}^{(N)} = \sum_{J,K=1}^{N+1} \left(-\Theta(J-K)|JK\rangle\langle JK| + \Theta(J-K)|KJ\rangle\langle JK|\right). \tag{5}$$

The Markov matrix has zero as an eigenvalue and the corresponding eigenvector is called the stationary state. For a given number $m_i \in \mathbb{N}$ of particles of type i (with $1 \leq i \leq N+1$), the stationary state is unique. All the other eigenvalues have strictly negative real parts, which characterize the relaxation to the stationary states. The

Markov matrix defines an integrable model that can be solved by means of the nested algebraic Bethe Ansatz [1, 5]. Besides, in [5], the spectral structure of the N-ASEP Markov matrix was investigated; one simple result is that the spectrum of the Markov matrix of N species of particles contains the spectra of systems with $N' < N$ species.

2.2. Matrix Ansatz for the stationary state

Although the N-ASEP is integrable by Bethe Ansatz, the explicit calculation of eigenvectors is not easy even for the stationary states. An alternative technique for studying the stationary state is the matrix product Ansatz (which we call simply matrix Ansatz), first used in [13]. This trick has grown into very powerful method to analyze one-dimensional systems out of equilibrium [7, 11]. The matrix Ansatz for the stationary state of the N-ASEP was constructed in [14, 24]. The basic idea is to write the stationary weight for a configuration $k_1 \cdots k_L$ as a trace of a matrix product

$$P(k_1 \cdots k_L) = \frac{1}{Z} \operatorname{Tr}(X_{k_1} \cdots X_{k_L}), \tag{6}$$

where the operator X_k corresponds to the k-th species and Z is a normalization factor. The operators $X_k (k = 1, \ldots, N+1)$ must obey specific relations in order for the expression (6) to represent the stationary state. We emphasize that the representation space of these operators X_k is not the physical space but an abstract vector space which is usually infinite dimensional. It was shown in [14, 24] that the X_k's can be chosen as sums of tensor products of $\delta_q, \epsilon_q, A_q$ and $\mathbb{1}$, which satisfy the relations

$$\delta_q \epsilon_q - q \epsilon_q \delta_q = (1 - q)\mathbb{1}, \quad \delta_q A_q = q A \delta_q, \quad A_q \epsilon_q = q \epsilon_q A_q. \tag{7}$$

This algebra is related to the quantum harmonic oscillator and the quantum group [3, 22, 27, 30]. An explicit representation of this quadratic algebra is given, for example, by the following infinite dimensional matrices [13]:

$$\delta_q = \begin{pmatrix} 0 & c_1 & & \\ & 0 & c_2 & \\ & & 0 & \ddots \\ & & & \ddots \end{pmatrix}, \quad \epsilon_q = \begin{pmatrix} 0 & & & \\ c_1 & 0 & & \\ & c_2 & 0 & \\ & & \ddots & \ddots \end{pmatrix}, \quad A_q = \begin{pmatrix} 1 & & & \\ & q & & \\ & & q^2 & \\ & & & \ddots \end{pmatrix}, \tag{8}$$

and $\mathbb{1} = A_1$ with $c_i = \sqrt{1 - q^i}$.

2.3. Poset structure and spectral inclusion

Since the number of each class of particles is conserved, the total Markov matrix $M^{(N)}$ (4) splits into blocks as

$$M^{(N)} = \bigoplus_m M_m, \quad \text{with} \quad M_m \in \operatorname{End} V_m, \quad \text{where} \quad V_m = \bigoplus_{\#\{i|k_i=j\}=m_j} \mathbb{C}|k_1 \cdots k_L\rangle. \tag{9}$$

We labeled each diagonal block M_m and the corresponding vector space (sector) V_m by $m = (m_1, \ldots, m_{N+1})$. We will call the label itself a sector, and in particular, a *basic sector* corresponds to the case $m_i > 0$ for all i. A useful alternative labeling of the basic sectors [5] is obtained as follows: let s_j be the total number of particles of classes $k \leq j$,

$$s_j = m_1 + m_2 + \cdots + m_j. \tag{10}$$

One has $m_j = s_j - s_{j-1} > 0$ ($s_0 = 0$), and thus each basic sector can be labeled by the set $\mathfrak{s} = \{s_1, \ldots, s_N\} \subset \{1, 2, \ldots, L-1\} = \Omega$ with $0 < s_1 < s_2 < \cdots < s_N < L$. The set

$\mathfrak{s}$ is an element of $\mathcal{S}$, the power set (the set of all subsets) of Ω. In the following, we shall use both labels equivalently: for instance, the invariant vector spaces (respectively the Markov matrices acting on them) will be denoted either by V_m or $V_{\mathfrak{s}}$ (respectively M_m or $M_{\mathfrak{s}}$). The set $\mathcal{S}$ is equipped with a natural poset (partially ordered set) structure with respect to the inclusion $\subseteq$, which is encoded in the Hasse diagram. In our case it is simply the $L - 1$ dimensional hypercube where each vertex of the hypercube corresponds to a sector, and each edge corresponds to an arrow $\mathfrak{t} \to \mathfrak{s}$ meaning that $\mathfrak{t} \subset \mathfrak{s}$ and $\#\mathfrak{s} = \#\mathfrak{t} + 1$. (See figure 1.)
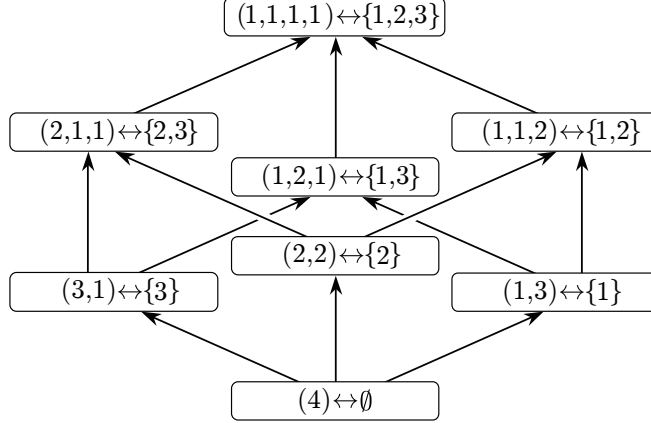

Figure 1. The Hasse diagram for $L = 4$. Each basic sector is labelled in two ways.

The spectral properties of the Markov matrix on the Hasse diagram were investigated in [5]. Given two sectors $\mathfrak{s} = \{s_1 < \cdots < s_N\}$ and $\mathfrak{t} = \mathfrak{s} \setminus \{s_{n_1}, \ldots, s_{n_u}\}$ connected by a finite sequence of arrows in the Hasse diagram, one could define an identification operator from $\mathfrak{s}$ to $\mathfrak{t}$ as follows

$$\varphi_{\mathfrak{t}\mathfrak{s}} : \ |k_1 \cdots k_L\rangle \in V_{\mathfrak{s}} \ \mapsto \ |k_1' \cdots k_L'\rangle \in V_{\mathfrak{t}} \quad \text{with} \quad x' = x - \#\{i | n_i < x\}. \tag{11}$$

Using $\varphi_{\mathfrak{t}\mathfrak{s}}$, a *conjugation relation* could be proved: $\varphi_{\mathfrak{t}\mathfrak{s}} M_{\mathfrak{s}} = M_{\mathfrak{t}} \varphi_{\mathfrak{t}\mathfrak{s}}$. This relation implies that by applying $\varphi_{\mathfrak{t}\mathfrak{s}}$, an eigenvector $|E\rangle_{\mathfrak{s}}$ of the sector $\mathfrak{s}$ with an eigenvalue E is either projected to an eigenvector $\varphi_{\mathfrak{t}\mathfrak{s}}|E\rangle_{\mathfrak{s}}$ in sector $\mathfrak{t}$ or is killed out. The surjectivity of $\varphi_{\mathfrak{t}\mathfrak{s}}$ leads to the spectral inclusion $\mathrm{Spec}(M_{\mathfrak{s}}) \supset \mathrm{Spec}(M_{\mathfrak{t}})$.

2.4. Generalized matrix Ansatz and the hat-algebra

It should be noted that the action of $\varphi_{\mathfrak{t}\mathfrak{s}}$ loses information by projecting a larger sector $\mathfrak{s}$ into a smaller one $\mathfrak{t}$. What is desirable is a *conjugation operator* (*conjugation matrix*)

$\psi_{\mathfrak{s}\mathfrak{t}} : V_{\mathfrak{t}} \to V_{\mathfrak{s}}$ satisfying the opposite conjugation relation

$$M_{\mathfrak{s}}\psi_{\mathfrak{s}\mathfrak{t}} = \psi_{\mathfrak{s}\mathfrak{t}}M_{\mathfrak{t}}\,, \tag{12}$$

which can be expressed by the following commutative diagram:

$$
\begin{array}{ccc}
V_{\mathfrak{s}} & \xrightarrow{M_{\mathfrak{s}}} & V_{\mathfrak{s}} \\
\psi_{\mathfrak{s}\mathfrak{t}}\uparrow & & \uparrow\psi_{\mathfrak{s}\mathfrak{t}} \\
V_{\mathfrak{t}} & \xrightarrow{M_{\mathfrak{t}}} & V_{\mathfrak{t}}
\end{array}
\tag{13}
$$

The form for $\psi_{\mathfrak{s}\mathfrak{t}}$ is nontrivial in general, and its action helps us to construct eigenvectors (including stationary states) from lower sectors.

In [4] a method to construct the conjugation matrix $\psi_{\mathfrak{s}\mathfrak{t}}$ was introduced by generalizing the matrix Ansatz for the stationary state. The basic idea is to write each element of $\psi_{\mathfrak{s}\mathfrak{t}}$ in a form

$$\langle j_1 \cdots j_L | \psi_{\mathfrak{s}\mathfrak{t}} | k_1 \cdots k_L \rangle = \mathrm{Tr}(a_{j_1 k_1} \cdots a_{j_L k_L}) \tag{14}$$

with matrices a_{JK} $(1 \le J \le N+1, 1 \le K \le N)$.

Consider for a nearest-neighbor pair of sectors $\mathfrak{s} = \{s_1 < \cdots < s_N\}$ of N-species and $\mathfrak{t} = \mathfrak{s} \setminus \{s_n\}$ of $(N-1)$-species. The sector $\mathfrak{t}$ is obtained from the sector $\mathfrak{s}$ by merging the particles of nth and $(n+1)$st classes into the same class. It will be convenient to write these operators as elements of an operator-valued matrix $\mathbf{a}$ such that

$$a_{JK} = \langle J | \mathbf{a} | K \rangle\,. \tag{15}$$

Following [4], one can prove that the matrix $\psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_n\}}$ defined by equation (14) is a conjugation operator if there exists another operator-valued matrix $\widehat{\mathbf{a}}^{(N,n)}$ of size $N+1$ by N that satisfies the following key relation (*hat relation*):

$$M_{\mathrm{Loc}}^{(N)}(\mathbf{a}\otimes\mathbf{a}) - (\mathbf{a}\otimes\mathbf{a})M_{\mathrm{Loc}}^{(N-1)} = \mathbf{a}\otimes\widehat{\mathbf{a}} - \widehat{\mathbf{a}}\otimes\mathbf{a}\,. \tag{16}$$

The elements of this equation give a set of relations, which we call the *hat algebra*. We emphasize that the hat relation (16) connects the local Markov matrix of the N-ASEP and that of the $(N-1)$-ASEP, and is of a different nature from the relations used in [17, 26] for the usual matrix Ansatz for the stationary state.

Therefore, the problem of constructing a conjugation operator has been reduced to finding realizations of the hat algebra.

2.5. The N-TASEP case

In the TASEP case, a family of solutions $(\mathbf{a}, \widehat{\mathbf{a}}) = (\mathbf{a}^{(N,n)}, \widehat{\mathbf{a}}^{(N,n)})$ $(1 \le n \le N)$ to the hat relation (16) was successfully constructed in [4], where n is the class being split. The operator-valued matrices $\mathbf{a}^{(N,n)}$ and $\widehat{\mathbf{a}}^{(N,n)}$ are related by multiplication with a diagonal matrix:

$$\widehat{\mathbf{a}}^{(N,n)} = \mathrm{diag}(\underbrace{1,\ldots,1}_{n},\underbrace{0,\ldots,0}_{N+1-n})\mathbf{a}^{(N,n)}\,. \tag{17}$$

Then the hat algebra (16) becomes a closed quadratic algebra generated by the elements of $\mathbf{a}^{(N,n)}$. An explicit representation of the hat algebra is constructed by $(N-1)$-fold tensor products of the fundamental quadratic algebra generated by the infinite dimensional matrices δ, ϵ and A that satisfy the relations [13]

$$\delta\,\epsilon = 1\,, \quad \delta\,A = 0\,, \quad A\,\epsilon = 0\,. \tag{18}$$

We note that they are obtained by taking $q = 0$ in (8).

The expressions of the operators $a_{JK}^{(N,n)}$ are given in the following table with $\delta\otimes\mathbb{1}^{\otimes(-1)}\otimes\epsilon = \mathbb{1}$ and $\delta\otimes\mathbb{1}^{\otimes x}\otimes\epsilon = 0$ for $x \leq -2$.

$J\backslash K$	$1 \quad \cdots \quad n-1$	n	$n+1 \quad \cdots \quad N$
1 $\vdots$ $n-1$	$A^{\otimes(J-1)}\otimes\delta\otimes\mathbb{1}^{\otimes(K-J-1)}$ $\otimes\epsilon\otimes\mathbb{1}^{\otimes(N-K-1)}$	$A^{\otimes(J-1)}\otimes$ $\delta\otimes\mathbb{1}^{\otimes(N-J-1)}$	$A^{\otimes(J-1)}\otimes\delta\otimes\mathbb{1}^{\otimes(K-J-2)}$ $\otimes\delta\otimes\mathbb{1}^{\otimes(N-K)}$
n	0	$A^{\otimes(n-1)}\otimes$ $\mathbb{1}^{\otimes(N-n)}$	$A^{\otimes(n-1)}\otimes\mathbb{1}^{\otimes(K-n-1)}$ $\otimes\delta\otimes\mathbb{1}^{\otimes(N-K)}$
$n+1$	$\mathbb{1}^{\otimes(K-1)}\otimes\epsilon\otimes$ $\mathbb{1}^{\otimes(n-K-1)}\otimes A^{\otimes(N-n)}$	$\mathbb{1}^{\otimes(n-1)}\otimes$ $A^{\otimes(N-n)}$	0
$n+2$ $\vdots$ $N+1$	$\mathbb{1}^{\otimes(K-1)}\otimes\epsilon\otimes\mathbb{1}^{\otimes(J-K-3)}$ $\otimes\epsilon\otimes A^{\otimes(N-J+1)}$	$\mathbb{1}^{\otimes(J-3)}\otimes$ $\epsilon\otimes A^{\otimes(N-J+1)}$	$\mathbb{1}^{\otimes(K-2)}\otimes\delta\otimes\mathbb{1}^{\otimes(J-K-2)}$ $\otimes\epsilon\otimes A^{\otimes(N-J+1)}$

$$(19)$$

3. Generalized matrix Ansatz for the N-PASEP

In this section, we explain how to construct representations of the hat algebra and the corresponding conjugation operator for the N-PASEP case. It turns out that finding a representation for the hat algebra of the N-PASEP is not a simple deformation of the N-TASEP case. It will require a construction more involved and rather different from that used for the N-TASEP.

3.1. The hat relations for PASEP

For any given value of N, the TASEP solution (19) can be generalized to the PASEP in the two special cases $n = 1$ and $n = N$. The case $n = 1$ corresponds to splitting the first-class particles in two sub-classes. The dual case $n = N$ corresponds to splitting the holes (labelled as N's) in the $(N - 1)$-PASEP model into Nth-class particles and holes (now labelled as $(N + 1)$'s) in the N-PASEP model. The generalization is obtained by the replacement $\{\delta, \epsilon, A\} \to \{\delta_q, \epsilon_q, A_q\}$ (8) in each element of $\mathbf{a}^{(N,n)}$ and a simple modification of (17)

$$\widehat{\mathbf{a}}^{(N,1)} = \mathrm{diag}(1, q, \ldots, q)\, \mathbf{a}^{(N,1)}, \tag{20}$$

$$\widehat{\mathbf{a}}^{(N,N)} = \mathrm{diag}(1, \ldots, 1, q)\, \mathbf{a}^{(N,N)}, \tag{21}$$

so that the hat relation (16) is satisfied. The hat algebra of the TASEP case is deformed by this modification.

For general n (with $1 < n < N$), equations (20) and (21) naturally lead us to assume the following relation between $\widehat{\mathbf{a}}^{(N,n)}$ and $\mathbf{a}^{(N,n)}$:

$$\widehat{\mathbf{a}}^{(N,n)} = d^{(N,n)}\mathbf{a}^{(N,n)} = \mathrm{diag}(\underbrace{1, \ldots, 1}_{n}, \underbrace{q, \ldots, q}_{N+1-n})\, \mathbf{a}^{(N,n)}. \tag{22}$$

Inserting this equation in the hat relation (16), one finds a *closed* quadratic algebra for the operators a_{JK}.

The quadratic hat algebra for the N-PASEP is defined by a set of relations between its generators, summarized as‡

$$
\begin{array}{|c|c|c|}
\hline
 & \mathrm{I}: K < K' & \mathrm{II}: K \geq K' \\
\hline
\mathrm{A}: J \leq n < J' & \begin{array}{c} a_{JK'}a_{J'K} - qa_{J'K}a_{JK'} \\ = (1-q)a_{JK}a_{J'K'} \end{array} & qa_{J'K}a_{JK'} = qa_{JK'}a_{J'K} \\
\hline
\mathrm{B}: \begin{array}{c} J < J' \leq n \ \text{or} \\ n < J < J' \end{array} & a_{JK'}a_{J'K} = qa_{J'K}a_{JK'} & \begin{array}{c} qa_{J'K}a_{JK'} - qa_{JK'}a_{J'K} \\ = (1-q)a_{JK}a_{J'K'} \end{array} \\
\hline
\mathrm{C}: J = J' & a_{JK}a_{JK'} = a_{JK'}a_{JK} & qa_{JK}a_{JK'} = qa_{JK'}a_{JK} \\
\hline
\mathrm{D}: J > n \geq J' & a_{J'K}a_{JK'} = a_{JK'}a_{J'K} & \begin{array}{c} a_{J'K}a_{JK'} - qa_{JK'}a_{J'K} \\ = (1-q)a_{JK}a_{J'K'} \end{array} \\
\hline
\mathrm{E}: \begin{array}{c} J > J' > n \ \text{or} \\ n \geq J > J' \end{array} & \begin{array}{c} a_{JK'}a_{J'K} - a_{J'K}a_{JK'} \\ = (1-q)a_{JK}a_{J'K'} \end{array} & \begin{array}{c} a_{J'K}a_{JK'} \\ = qa_{JK'}a_{J'K} \end{array} \\
\hline
\end{array}
\tag{23}
$$

Our aim is to construct an explicit representation for this N-PASEP algebra. At first thought, one would expect that the replacement $\{\delta, \epsilon, A\} \to \{\delta_q, \epsilon_q, A_q\}$ in the TASEP solution (19) provides us with a representation of the hat algebra (23) for any value of n. Unfortunately, this is true only for $n = 1$ and $n = N$. For general n $(1 < n < N)$, we found no perturbative representation $a_{JK}^{(N,n)} = a_{JK}^{(N,n)}\big|_{\mathrm{TASEP}} + qb_{JK}^{(N,n)} + q^2 c_{JK}^{(N,n)} + \cdots$ for the algebra (23) starting from the TASEP representation (19). The heuristic reason behind this fact can be summarized as follows. If one considers the fundamental TASEP algebra $\mathcal{T}$ generated by δ, ϵ and A then the tensor products $\delta \otimes \delta$, $\epsilon \otimes \epsilon$ and $A \otimes A$ also generate the same algebra i.e. they satisfy the same quadratic relations. Therefore there exists a simple coproduct operation from $\mathcal{T}$ to $\mathcal{T} \otimes \mathcal{T}$ that preserves the algebraic structure (18) [9]. However, one can easily verify that $\delta_q \otimes \delta_q$, $\epsilon_q \otimes \epsilon_q$ and $A_q \otimes A_q$ do not satisfy the fundamental PASEP relations (7). It seems that no coproduct exists for PASEP algebra that would allow us to build natural tensor representations. This is the mathematical obstruction that prevents us from constructing the N-PASEP hat algebra by deforming the known N-TASEP hat algebra.

To summarize, the solution for the N-TASEP (19) is of no help in general to find representations of the N-PASEP algebra (23). One needs a different strategy to build explicit representations, which we will explain beginning with a specific example in the next subsection.

3.2. The simplest non-trivial example

In this subsection, we work out the simplest non-trivial example with $1 < n < N$: this is obtained for the case $(N, n) = (3, 2)$. In particular, we show that the relations (23) are not contradictory by giving an explicit representation. Hence they define a bona fide algebra.

A solution to (23) will be constructed by using the regular representation of this algebra. We assume that

$$
a_{11} = a_{43} = Id, \quad a_{21} = a_{33} = 0,
\tag{24}
$$

‡ The cases A-II and C-II vanish when $q = 0$.

PASEP	TASEP	Ordering
$a_{13}a_{12} = a_{12}a_{13}$	$a_{13}a_{12} = a_{12}a_{13}$	
$a_{12}a_{22} = qa_{22}a_{12}$	$a_{12}a_{22} = 0$	$a_{12} \prec a_{22}$
$a_{13}a_{22} = qa_{22}a_{13}$	$a_{13}a_{22} = 0$	$a_{13} \prec a_{22}$
$a_{23}a_{12} = a_{12}a_{23} + (1-q)a_{22}a_{13}$	$a_{23}a_{12} = a_{12}a_{23} + a_{22}a_{13}$	
$a_{13}a_{23} = qa_{23}a_{13}$	$a_{13}a_{23} = 0$	$a_{13} \prec a_{23}$
$a_{23}a_{22} = a_{22}a_{23}$	$a_{23}a_{22} = a_{22}a_{23}$	
$a_{12}a_{31} = qa_{31}a_{12} + (1-q)a_{32}$	$a_{12}a_{31} = a_{32}$	$a_{12} \prec a_{31}$
$a_{13}a_{31} = qa_{31}a_{13}$	$a_{13}a_{31} = 0$	$a_{13} \prec a_{31}$
$a_{22}a_{31} = qa_{31}a_{22}$	$a_{22}a_{31} = 0$	$a_{22} \prec a_{31}$
$a_{23}a_{31} = qa_{31}a_{23}$	$a_{23}a_{31} = 0$	$a_{23} \prec a_{31}$
$a_{12}a_{32} = a_{32}a_{12}$	$a_{12}a_{32} = a_{32}a_{12}$	
$a_{13}a_{32} = qa_{32}a_{13}$	$a_{13}a_{32} = 0$	$a_{13} \prec a_{32}$
$a_{22}a_{32} = a_{32}a_{22}$	$a_{22}a_{32} = a_{32}a_{22}$	
$a_{23}a_{32} = qa_{32}a_{23}$	$a_{23}a_{32} = 0$	$a_{23} \prec a_{32}$
$a_{32}a_{31} = a_{31}a_{32}$	$a_{32}a_{31} = a_{31}a_{32}$	
$a_{12}a_{41} = qa_{41}a_{12} + (1-q)a_{42}$	$a_{12}a_{41} = a_{42}$	$a_{12} \prec a_{41}$
$a_{13}a_{41} = qa_{41}a_{13} + (1-q)Id$	$a_{13}a_{41} = Id$	$a_{13} \prec a_{41}$
$a_{22}a_{41} = qa_{41}a_{22}$	$a_{22}a_{41} = 0$	$a_{22} \prec a_{41}$
$a_{23}a_{41} = qa_{41}a_{23}$	$a_{23}a_{41} = 0$	$a_{23} \prec a_{41}$
$a_{31}a_{41} = qa_{41}a_{31}$	$a_{31}a_{41} = 0$	$a_{31} \prec a_{41}$
$a_{32}a_{41} = qa_{41}a_{32}$	$a_{32}a_{41} = 0$	$a_{32} \prec a_{41}$
$a_{12}a_{42} = a_{42}a_{12}$	$a_{12}a_{42} = a_{42}a_{12}$	
$a_{13}a_{42} = qa_{42}a_{13} + (1-q)a_{12}$	$a_{13}a_{42} = a_{12}$	$a_{13} \prec a_{42}$
$a_{22}a_{42} = a_{42}a_{22}$	$a_{22}a_{42} = a_{42}a_{22}$	
$a_{23}a_{42} = qa_{42}a_{23} + (1-q)a_{22}$	$a_{23}a_{42} = a_{22}$	$a_{23} \prec a_{42}$
$a_{42}a_{31} = a_{31}a_{42} + (1-q)a_{41}a_{32}$	$a_{42}a_{31} = a_{31}a_{42} + a_{41}a_{32}$	
$a_{32}a_{42} = qa_{42}a_{32}$	$a_{32}a_{42} = 0$	$a_{32} \prec a_{42}$
$a_{42}a_{41} = a_{41}a_{42}$	$a_{42}a_{41} = a_{41}a_{42}$	

Table 1. The hat algebra (23) for $(N,n) = (3,2)$. The left and middle columns correspond to the PASEP and TASEP cases, respectively. In the right column the ordering restricted by the algebra of the TASEP case.

as in the TASEP solution. We also assume that equation (22) is valid so that $\widehat{\mathbf{a}}^{(3,2)} = (1,1,q,q)\,\mathbf{a}^{(3,2)}$. Inserting these assumptions in the hat relation (16), one obtains 28 relations, shown in the leftmost column of table 1, that have to be satisfied by the 8 unknown a_{ij}'s. Let us consider the space of monomials generated by a_{ij}'s. The TASEP algebra (the middle column of table 1) tells us the correct order of the unknowns a_{ij} in an arbitrary word consisting of these monomials.

For example, a_{12} should be located to the right of a_{22} ($a_{22} \succ a_{12}$) due to the relation $a_{12}a_{22} = 0$. We listed the restriction for the reordering of the generators in the right column of table 1. One of the allowed orderings is

$$a_{41} \succ a_{31} \succ a_{42} \succ a_{32} \succ a_{22} \succ a_{12} \succ a_{23} \succ a_{13}. \tag{25}$$

Let us consider the right-action of each generator a_{ij} on the monomial (word)

$$W = a_{41}^{n_{41}} a_{31}^{n_{31}} \cdots a_{13}^{n_{13}} \quad (n_{ij} \geq 0), \tag{26}$$

for the TASEP case. We set

$$a_{13} = \mathbb{1}^{\otimes 7} \otimes \delta. \tag{27}$$

From the relation $a_{13}a_{23} = 0$, one possibility for a_{23} is

$$a_{23} = \mathbb{1}^{\otimes 6}\otimes\delta\otimes A. \tag{28}$$

From the first, forth and sixth relations, the action of a_{12} is calculated as

$$W a_{12} = \begin{cases} a_{41}^{n_{41}}\cdots a_{22}^{n_{22}}a_{12}a_{23}^{n_{23}}a_{13}^{n_{13}} + a_{41}^{n_{41}}\cdots a_{32}^{n_{32}}a_{22}^{n_{22}+1}a_{23}^{n_{23}-1}a_{13}^{n_{13}} & (n_{12} = 0) \\ a_{41}^{n_{41}}\cdots a_{22}^{n_{22}}a_{12}^{n_{12}+1}a_{23}^{n_{23}}a_{13}^{n_{13}} & (n_{12} \geq 1) \end{cases} \tag{29}$$

Thus one can set

$$a_{12} = \mathbb{1}^{\otimes 5}\otimes\delta\otimes\mathbb{1}^{\otimes 2} + \mathbb{1}^{\otimes 4}\otimes\delta\otimes A\otimes\epsilon\otimes\delta. \tag{30}$$

In a similar way, we obtain conditions for the other monomials in the order $a_{22},\ldots,a_{41}$, and we end up with

$$a_{22} = \mathbb{1}^{\otimes 4}\otimes\delta\otimes A\otimes\mathbb{1}\otimes A, \quad a_{32} = \mathbb{1}^{\otimes 3}\otimes\delta\otimes\mathbb{1}^{\otimes 2}\otimes A^{\otimes 2}, \tag{31}$$

$$a_{42} = \mathbb{1}^{\otimes 5}\otimes\delta\otimes\mathbb{1}\otimes\epsilon + \mathbb{1}^{\otimes 4}\otimes\delta\otimes A\otimes\epsilon\otimes\mathbb{1} + \mathbb{1}^{\otimes 2}\otimes\delta\otimes A\otimes\mathbb{1}^{\otimes 2}\otimes A^{\otimes 2}, \tag{32}$$

$$a_{31} = \mathbb{1}\otimes\delta\otimes\mathbb{1}^{\otimes 2}\otimes A^{\otimes 4} + \delta\otimes A\otimes\epsilon\otimes\delta\otimes A^{\otimes 4} + \mathbb{1}^{\otimes 3}\otimes\delta\otimes\mathbb{1}\otimes\epsilon\otimes A^{\otimes 2}, \tag{33}$$

$$a_{41} = \mathbb{1}^{\otimes 7}\otimes\epsilon + \mathbb{1}^{\otimes 2}\otimes\delta\otimes A\otimes\mathbb{1}\otimes\epsilon\otimes A^{\otimes 2} + \delta\otimes A\otimes\mathbb{1}\otimes A^{\otimes 5}. \tag{34}$$

Accidentally (and fortunately), the q-deformation of δ's, ϵ's and A's in a_{ij}'s gives a solution to the hat relation for general q. Furthermore, the following simplification still keeps the hat relation satisfied. Erase the 1st, 2nd, 3rd, 4th and 5th components of the tensor products in each a_{ij}. Erase the 3rd term in a_{42}, the 1st and 2nd terms in a_{31} and the 2nd and 3rd terms in a_{41}. Then we get a solution

$$\mathbf{a}^{(3,2)} = \begin{matrix} & \overset{1}{} & \overset{2}{} & \overset{3}{} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \left(\begin{matrix} \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1} & A\otimes\epsilon\otimes\delta + \delta\otimes\mathbb{1}\otimes\mathbb{1} & \mathbb{1}\otimes\mathbb{1}\otimes\delta \\ 0 & A\otimes\mathbb{1}\otimes A & \mathbb{1}\otimes\delta\otimes A \\ \epsilon\otimes A\otimes A & \mathbb{1}\otimes A\otimes A & 0 \\ \mathbb{1}\otimes\mathbb{1}\otimes\epsilon & A\otimes\epsilon\otimes\mathbb{1} + \delta\otimes\mathbb{1}\otimes\epsilon & \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1} \end{matrix}\right) \end{matrix}. \tag{35}$$

It is straightforward to verify that the a_{JK}'s satisfy the algebra (23). Using this representation, one can calculate all matrix elements (14) of the conjugation operator $\psi_{\{1,2,3\},\{1,3\}}$ between the sectors $\mathfrak{s} = \{1,2,3\}$ and $\mathfrak{t} = \{1,3\}$. For example, one has

$$\begin{aligned}
\langle 1324|\psi_{\{1,2,3\},\{1,3\}}|2132\rangle &= \mathrm{Tr}\left(a_{12}^{(3,2)}a_{31}^{(3,2)}a_{23}^{(3,2)}a_{42}^{(3,2)}\right) \\
&= \mathrm{Tr}\left((A\otimes\epsilon\otimes\delta + \delta\otimes\mathbb{1}\otimes\mathbb{1})(\epsilon\otimes A\otimes A)(\mathbb{1}\otimes\delta\otimes A)(A\otimes\epsilon\otimes\mathbb{1} + \delta\otimes\mathbb{1}\otimes\epsilon)\right) \\
&= \mathrm{Tr}(A\epsilon A)\,\mathrm{Tr}(\epsilon A\delta\epsilon)\,\mathrm{Tr}(\delta A^2) + \mathrm{Tr}(A\epsilon\delta)\,\mathrm{Tr}(\epsilon A\delta)\,\mathrm{Tr}(\delta A^2\epsilon) \\
&\quad + \mathrm{Tr}(\delta\epsilon A)\,\mathrm{Tr}(A\delta\epsilon)\,\mathrm{Tr}(A^2) + \mathrm{Tr}(\delta\epsilon\delta)\,\mathrm{Tr}(A\epsilon)\,\mathrm{Tr}(A^2\epsilon) \\
&= \frac{1 + q^2}{(1 - q^2)^2(1 - q^3)}.
\end{aligned} \tag{36}$$

The full elements of matrix $\psi_{\{1,2,3\},\{1,3\}}$ are given by:

$$\psi_{\{1,2,3\},\{1,3\}} = \begin{pmatrix}
a & \cdot & qb & \cdot & \cdot & qb & \cdot & cq & qd^3 & \cdot & qd^3 & q^2e \\
\cdot & a & b & \cdot & \cdot & \cdot & qb & q^2d & qe & \cdot & qe & qd \\
a & b & \cdot & b & c & \cdot & q^2d & \cdot & \cdot & q^2d & \cdot & q^2e \\
qb & a & \cdot & qe & b & qd & \cdot & \cdot & \cdot & q^2d & qe & \cdot \\
\cdot & qb & a & \cdot & qd & \cdot & cq & \cdot & qb & e & \cdot & qd \\
b & \cdot & a & d & \cdot & c & \cdot & b & \cdot & e & d & \cdot \\
\cdot & \cdot & \cdot & a & \cdot & b & \cdot & q^2d & qe & qb & qe & qd \\
\cdot & \cdot & \cdot & \cdot & a & \cdot & b & e & d & b & d & c \\
\cdot & \cdot & \cdot & qb & qd & a & e & \cdot & \cdot & cq & qb & qd \\
\cdot & \cdot & \cdot & qd^3 & qb & q^2e & a & \cdot & \cdot & cq & qd^3 & qb \\
\cdot & \cdot & \cdot & \cdot & q^2e & \cdot & q^2d & a & b & q^2d & b & c \\
\cdot & \cdot & \cdot & qe & \cdot & qd & \cdot & qb & a & q^2d & qe & b \\
qb & qe & qd & a & b & \cdot & q^2d & \cdot & qe & \cdot & \cdot & \cdot \\
cq & qb & qd & qb & a & qd & \cdot & e & \cdot & \cdot & \cdot & \cdot \\
b & d & c & \cdot & \cdot & a & e & b & d & \cdot & \cdot & \cdot \\
q^2d & b & c & \cdot & \cdot & q^2e & a & q^2d & b & \cdot & \cdot & \cdot \\
cq & qd^3 & qb & qd^3 & q^2e & qb & \cdot & a & \cdot & \cdot & \cdot & \cdot \\
q^2d & qe & b & qe & qd & \cdot & qb & \cdot & a & \cdot & \cdot & \cdot \\
\cdot & qd^3 & q^2e & \cdot & qb & \cdot & cq & \cdot & qd^3 & a & \cdot & qb \\
q^2d & \cdot & q^2e & b & \cdot & c & \cdot & q^2d & \cdot & a & b & \cdot \\
\cdot & qe & qd & \cdot & \cdot & \cdot & q^2d & qb & qe & \cdot & a & b \\
e & \cdot & qd & \cdot & \cdot & qd & \cdot & cq & qb & \cdot & qb & a \\
q^2d & qe & \cdot & qe & qd & b & \cdot & \cdot & \cdot & qb & a & \cdot \\
e & d & \cdot & d & c & \cdot & b & \cdot & \cdot & b & \cdot & a
\end{pmatrix}, \tag{37}$$

where we have set

$$a = \frac{1}{(1-q)^2(1-q^2)}, \quad b = \frac{1}{(1-q)(1-q^2)^2}, \quad c = \frac{1+q^2}{(1-q^2)^2(1-q^3)}, \tag{38}$$

$$d = \frac{1}{(1-q^2)^2(1-q^3)}, \quad e = \frac{1}{(1-q)(1-q^2)(1-q^3)}, \tag{39}$$

and replaced 0 by a dot to make reading easier. The ordering of the bases is lexicographic: 1234,1243,...,4321 for the sector $\mathfrak{s} = \{1,2,3\}$, and 1223,1232,...,3221 for $\mathfrak{t} = \{1,3\}$. One can check explicitly that the conjugation relation $\psi_{\{1,2,3\},\{1,3\}} M_{\{1,3\}} = M_{\{1,2,3\}} \psi_{\{1,2,3\},\{1,3\}}$ is satisfied. We remark that all the nonzero elements of this example have a singularity at $q = 1$ of order 3.

3.3. The general case

One can construct a family of solutions to (16) for the general case (N, n) recursively by using the case $(N-1, n)$ if $n < N$ or the case $(N-1, n-1)$ if $n = N$:

$$\begin{array}{c}
\nwarrow \quad \nwarrow \quad \nwarrow \quad \nwarrow \quad \nearrow \\
(4,1) \quad (4,2) \quad (4,3) \quad (4,4) \\
\nwarrow \quad \nwarrow \quad \nwarrow \quad \nearrow \\
(3,1) \quad (3,2) \quad (3,3) \\
\nwarrow \quad \nwarrow \quad \nearrow \\
(2,1) \quad (2,2) \\
\nwarrow \quad \nearrow \\
(1,1)
\end{array} \tag{40}$$

More explicitly, for $1 \leq n \leq N-1$, $a^{(N,n)}$ is defined in terms of $a^{(N-1,n)}$ by the following formula:

$a_{JK}^{(N,n)} =$

$$
\begin{cases}
\displaystyle\sum_{J \le j \le n} a_{jK}^{(N-1,n)} \otimes \mathbb{1}^{\otimes(n-j)} \otimes \epsilon \otimes \mathbb{1}^{\otimes(j-1-J)} \otimes \delta \otimes A^{\otimes(J-1)} & (1 \le J \le n, 1 \le K \le N-1), \\
\mathbb{1}^{\otimes(E-J)} \otimes \delta \otimes A^{\otimes(J-1)} & (1 \le J \le n, K = N), \\
a_{JK}^{(N-1,n)} \otimes A^{\otimes n} & (n+1 \le J \le N, 1 \le K \le N-1), \\
0 & (n+1 \le J \le N, K = N), \\
\displaystyle\sum_{1 \le j \le n} a_{jK}^{(N-1,n)} \otimes \mathbb{1}^{\otimes(n-j)} \otimes \epsilon \otimes \mathbb{1}^{\otimes(j-1)} & (J = N+1, 1 \le K \le N-1), \\
\mathbb{1}^{\otimes E} & (J = N+1, K = N),
\end{cases}
\tag{41}
$$

where $E = E(N,n) = nN - n^2 + n - 1$ is the number of the tensor products, and $\epsilon \otimes \mathbb{1}^{\otimes(-1)} \otimes \delta = \mathbb{1}$. For $n = N$, $a^{(N,N)}$ is defined in terms of $a^{(N-1,N-1)}$ as

$$
a_{JK}^{(N,N)} =
\begin{cases}
\mathbb{1}^{\otimes(N-1)} & (J = K = 1), \\
\delta \otimes a_{N,K-1}^{(N-1,N-1)} & (J = 1, 2 \le K \le N), \\
0 & (2 \le J \le N, K = 1), \\
A \otimes a_{J-1,K-1}^{(N-1,N-1)} & (2 \le J \le N, 2 \le K \le N), \\
\epsilon \otimes \mathbb{1}^{\otimes(N-2)} & (J = N+1, K = 1), \\
\mathbb{1} \otimes a_{N,K-1}^{(N-1,N-1)} & (J = N+1, 2 \le K \le N).
\end{cases}
\tag{42}
$$

We remark that in this representation the number of tensor products grows typically as N^2 (supposing that n is of order N) whereas in the TASEP case the representation (19) involves only $(N-1)$-fold tensor products. Since the trace of each component of the tensor product gives the factor $\frac{1}{1-q}$, each nonzero element of ψ has a singularity at $q = 1$ of order $E(N,n)$.

This recursion has been obtained by investigating the case $(N,n) = (4,2)$ (see subsection 3.2) and by guessing the general (N,n) case. The general formulae (41) and (42) are proved by verifying that the quadratic relations (23) are satisfied. This is done by induction on the number of species N. I.e. one can check all the cases in the table (23) assuming these are satisfied for $N \to N-1$. The proof is not particularly illuminating, which is in the same spirit as the one given in [24].

When $q = 0$ and $n = 1, N$, the operators obtained from (41) and (42) are identical to the TASEP solution (19). This is not true anymore for $n \ne 1, N$. In other words, in the TASEP case $(q = 0)$, one has two different families of representations for the relations (23). We expect, however, that the conjugation matrices constructed by two different representations are identical:

$$
\psi_{\mathfrak{s},\mathfrak{s}\backslash\{s_n\}}|_{q=0} = \widetilde{\psi}_{\mathfrak{s},\mathfrak{s}\backslash\{s_n\}}
\tag{43}
$$

for $2 \le n \le N-1$ as well as for $n = 1, N$. So far this identity has been checked for small systems by using Mathematica.

3.4. Conjugation paths through the Hasse diagram

We finally consider the case of two PASEP models with respectively N and N' species and we suppose that N and N' are not consecutive integers. We show that a conjugation operator between the N-PASEP and the N'-PASEP can be constructed by using recursively the results of the previous sections.

First, we define a $\star$-product on operator valued matrices. Consider two operator valued matrices $\mathcal{A} = \{A_{ij}\}_{ij}$ and $\mathcal{B} = \{B_{ij}\}_{ij}$ and let the symbol $\star$ denote the product

$$\mathcal{A} \star \mathcal{B} = \{A_{ij} \otimes B_{jk}\}_{ik}, \tag{44}$$

which is bilinear and associative: $(\mathcal{A} \star \mathcal{B}) \star \mathcal{C} = \mathcal{A} \star (\mathcal{B} \star \mathcal{C})$. When $\mathcal{A}$ or $\mathcal{B}$ is a scalar-valued matrix, $\star$ is just the usual product. The following formula is satisfied:

$$(\mathcal{A} \otimes \mathcal{B}) \star (\mathcal{C} \otimes \mathcal{D}) = (\mathcal{A} \star \mathcal{C}) \otimes (\mathcal{B} \star \mathcal{D}). \tag{45}$$

We now suppose that we have found solutions for the hat algebra for all $N \in \mathbb{N}$:

$$M_{\text{Loc}}^{(N)} (\mathbf{a}^{(N)} \otimes \mathbf{a}^{(N)}) - (\mathbf{a}^{(N)} \otimes \mathbf{a}^{(N)}) M_{\text{Loc}}^{(N-1)} = \mathbf{a}^{(N)} \otimes \widehat{\mathbf{a}}^{(N)} - \widehat{\mathbf{a}}^{(N)} \otimes \mathbf{a}^{(N)}. \tag{46}$$

Then, using the $\star$-product one can construct a solution of the general hat-algebra defined as follows:

$$M_{\text{Loc}}^{(N)} (\mathcal{X} \otimes \mathcal{X}) = (\mathcal{X} \otimes \mathcal{X}) M_{\text{Loc}}^{(N')} + \mathcal{X} \otimes \widehat{\mathcal{X}} - \widehat{\mathcal{X}} \otimes \mathcal{X}. \tag{47}$$

Indeed, one can show by induction that this relation is satisfied by the choice

$$\mathcal{X} = \mathbf{a}^{(N)} \star \mathbf{a}^{(N-1)} \star \cdots \star \mathbf{a}^{(N'+1)}, \tag{48}$$

$$\widehat{\mathcal{X}} = \sum_{N \geq i \geq N'+1} \mathbf{a}^{(N)} \star \cdots \star \mathbf{a}^{(i+1)} \star \widehat{\mathbf{a}}^{(i)} \star \mathbf{a}^{(i-1)} \star \cdots \star \mathbf{a}^{(N'+1)}. \tag{49}$$

Generically there is no linear relation of the type (22) relating $\mathcal{X}$ and $\widehat{\mathcal{X}}$; hence, a closed algebra cannot be defined by the elements of $\mathcal{X}$ alone (except for the $N' = N-1$ case). Furthermore, we observe that for $1 \leq n < n' \leq N$, $a^{(N,n')} \star a^{(N-1,n)} \neq a^{(N,n)} \star a^{(N-1,n'-1)}$, and therefore $\mathcal{X}$ depends on the choice of the intermediate values of the n's. In other words, one has several solutions to (47). We conjecture, however, for a N-species sector $\mathfrak{s} = \{s_1, \ldots, s_N\}$ ($s_1 < \cdots < s_N, 1 \leq n < n' \leq N$), the "commutation relation" holds up to an overall factor as

$$\psi_{\mathfrak{s}, \mathfrak{s} \setminus \{s_n\}} \psi_{\mathfrak{s} \setminus \{s_n\}, \mathfrak{s} \setminus \{s_n, s_{n'}\}} = \frac{\prod_{i=n'+1}^{N} (1 - q^{s_i - s_n})}{\prod_{i=1}^{n-1} (1 - q^{s_{n'} - s_i})} \psi_{\mathfrak{s}, \mathfrak{s} \setminus \{s_{n'}\}} \psi_{\mathfrak{s} \setminus \{s_{n'}\}, \mathfrak{s} \setminus \{s_n, s_{n'}\}}, \tag{50}$$

or equivalently the following diagram commutes up to an overall factor:

$$
\begin{array}{ccc}
 & V_{\mathfrak{s}} & \\
\psi_{\mathfrak{s}, \mathfrak{s} \setminus \{s_n\}} \nearrow & & \nwarrow \psi_{\mathfrak{s}, \mathfrak{s} \setminus \{s_{n'}\}} \\
V_{\mathfrak{s} \setminus \{s_n\}} & & V_{\mathfrak{s} \setminus \{s_{n'}\}} \\
\psi_{\mathfrak{s} \setminus \{s_n\}, \mathfrak{s} \setminus \{s_n, s_{n'}\}} \nwarrow & & \nearrow \psi_{\mathfrak{s} \setminus \{s_{n'}\}, \mathfrak{s} \setminus \{s_n, s_{n'}\}} \\
 & V_{\mathfrak{s} \setminus \{s_n, s_{n'}\}} &
\end{array}
\tag{51}
$$

So far this identity has been checked for small systems by using Mathematica.

For $N' = 1$ the relation (47) corresponds to the usual matrix Ansatz for the stationary state (which is unique up to an overall constant in each sector), and one has several matrix representations for the stationary state according to the path $\emptyset \to \cdots \to \mathfrak{s}$ on the Hasse diagram. The stationary state was first obtained in [24] , where the path corresponds to $\emptyset \to \{s_1\} \to \{s_1, s_2\} \to \cdots \to \mathfrak{s} \setminus \{s_N\} \to \mathfrak{s}$.

Remark: In the symmetric exclusion case $p = q = 1$ (SSEP), the Markov matrix obeys the detailed-balance condition, and is a symmetric matrix: $M^{\text{T}} = M$. Thus the system in each sector converges to an equilibrium stationary state, where all the possible configurations are realized with an equal probability. Since the conjugation

matrix ψ has singularity $(1-q)^{-E(N,n)}$, we take the limit $q \to 1$ after multiplying it by $(1-q)^{E(N,n)}$:

$$\overline{\psi}_{\mathfrak{s},\mathfrak{s}\setminus\{s_n\}} = \lim_{q\to 1}(1-q)^{E(N,n)}\psi_{\mathfrak{s},\mathfrak{s}\setminus\{s_n\}} \tag{52}$$

for $\mathfrak{s} = \{s_1,\ldots,s_N\}$ and $1 \leq n \leq N$. This matrix satisfies the conjugation relation (12): $M_{\mathfrak{s}}\overline{\psi}_{\mathfrak{s},\mathfrak{s}\setminus\{s_n\}} = \overline{\psi}_{\mathfrak{s},\mathfrak{s}\setminus\{s_n\}}M_{\mathfrak{s},\setminus\{s_n\}}$. In the SSEP case, one can easily find a few other possibilities for a conjugation matrix satisfying (12). For example, the transpose of the identification operator φ also satisfies $M_{\mathfrak{s}}\varphi^{\mathrm{T}}_{\mathfrak{t}\mathfrak{s}} = \varphi^{\mathrm{T}}_{\mathfrak{t}\mathfrak{s}}M_{\mathfrak{t}}$ for $\mathfrak{s} \supset \mathfrak{t}$ since $M^{\mathrm{T}}_{\mathfrak{s}} = M_{\mathfrak{s}}$ and $M^{\mathrm{T}}_{\mathfrak{t}} = M_{\mathfrak{t}}$. Another simple solution (one dimensional representation) to the hat relation (16) with (22) is

$$a'_{JK} = \begin{cases} 0 & \begin{array}{l}(2 \leq J \leq n, 1 \leq K \leq J-1 \\ \quad \text{or } n+1 \leq J \leq N, J \leq K \leq N),\end{array} \\ 1 & \text{(otherwise)}. \end{cases} \tag{53}$$

Then the matrix ψ' defined as $\langle J_1 \cdots J_L|\psi'_{\mathfrak{s}\mathfrak{t}}|K_1 \cdots K_L\rangle = \prod_{1 \leq i \leq L} a'_{J_i K_i}$ also satisfies $M_{\mathfrak{s}}\psi'_{\mathfrak{s}\mathfrak{t}} = \psi'_{\mathfrak{s}\mathfrak{t}}M_{\mathfrak{t}}$.

4. The N-ASEP as a multistate vertex model

Let us consider the two dimensional vertex model, e.g. on $\ell \times L$ lattice. As in figure 2, each vertex (i,j) has a Boltzmann weight $W_{i,j}$ defined by values $(b_{i,j-1}, c_{i-1,j}, b_{i,j}, c_{i,j})$ that can be assigned to its four connected edges. The partition function is given by

$$Z = \sum_{\text{configuration}} \prod_{\text{vertices } (i,j)} W_{i,j}. \tag{54}$$

The Perk-Schultz model defines a family of vertex models that are exactly solvable [25]. As we will shortly review, the N-ASEP can be realized as a special case of the Perk-Schultz models for a particular choice of the vertex weights. We will also investigate the conjugation relation for this particular restriction of the Perk-Schultz model.

Define a matrix $R(\lambda) \in \mathrm{End}\left(\left(\mathbb{C}^{N+1}\right)^{\otimes 2}\right)$ as

$$R(\lambda) = \rho\left(1 + \lambda M^{(N)}_{\mathrm{Loc}}\right), \tag{55}$$

where ρ is the permutation matrix: $\rho(|\alpha\rangle\otimes|\beta\rangle) = |\beta\rangle\otimes|\alpha\rangle$. By regarding each element $R^{zw}_{xy}(\lambda) = \langle xy|R(\lambda)|zw\rangle$ as the Boltzmann weight of each vertex, the N-ASEP corresponds to a special case of the Perk-Schultz model with $N+1$ states, where the non-zero elements are given as

$$\begin{array}{ccc} \alpha & \alpha & \beta \\ \alpha \;\underset{\lambda}{+}\; \alpha & \alpha \;\underset{\lambda}{+}\; \beta & \alpha \;\underset{\lambda}{+}\; \alpha \\ \alpha & \beta & \beta \end{array} \tag{56}$$

$$R^{\alpha\alpha}_{\alpha\alpha}(\lambda) = 1, \quad R^{\alpha\beta}_{\alpha\beta}(\lambda) = \lambda\Theta(\alpha-\beta), \quad R^{\alpha\beta}_{\beta\alpha}(\lambda) = 1 - \lambda\Theta(\alpha-\beta),$$

see [5, 25, 29]. The matrix $R(\lambda)$ satisfies the Yang-Baxter relation

$$R_{bc}(\nu)R_{ac}(\mu)R_{ab}(\lambda) = R_{ab}(\lambda)R_{ac}(\mu)R_{bc}(\nu) \tag{57}$$

with $\lambda = \frac{\mu-\nu}{1-(p+q)\nu+pq\mu\nu}$. The indices a, b and c specify the spaces on which the R matrices act. The monodromy matrix is defined as

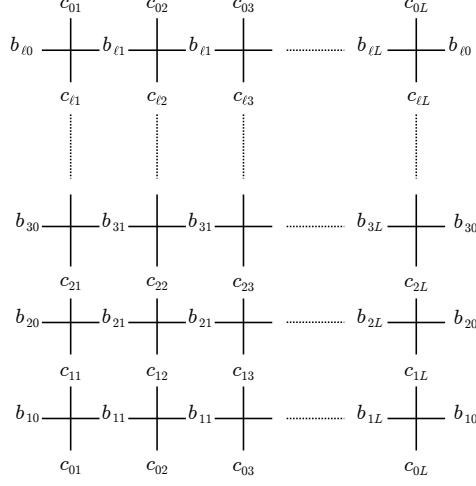$$\mathcal{T}_a(\lambda) = R_{aL}(\lambda)\cdots R_{a1}(\lambda) \tag{58}$$

Figure 2. The two-dimensional vertex model. The N-ASEP corresponds to the case where Boltzmann weights are given as equation (56).

acting on $\mathbb{C}^{N+1} \otimes \left(\mathbb{C}^{N+1}\right)^{\otimes L}$ where the first space is the so-called auxiliary space and is denoted by a. It satisfies the "global Yang-Baxter relation"

$$\mathcal{T}_b(\nu)\mathcal{T}_a(\mu)R_{ab}(\lambda) = R_{ab}(\lambda)\mathcal{T}_a(\mu)\mathcal{T}_b(\nu) \tag{59}$$

with $\lambda = \frac{\mu-\nu}{1-(p+q)\nu+pq\mu\nu}$. The row-to-row transfer matrix $T(\lambda)$ is defined by the trace of the monodromy matrix over the auxiliary space,

$$T(\lambda) = \mathrm{Tr}_a\mathcal{T}_a(\lambda). \tag{60}$$

The Yang-Baxter relation implies that it constitutes a one-parameter commuting family:

$$[T(\lambda_1), T(\lambda_2)] = 0. \tag{61}$$

The Markov matrix can be rewritten in terms of $T(\lambda)$ as

$$M^{(N)} = \left.\frac{\mathrm{d}}{\mathrm{d}\lambda} \log T(\lambda)\right|_{\lambda=0}, \tag{62}$$

which indeed satisfies $[M^{(N)}, T(\lambda)] = 0$. The transfer matrix $T(\lambda)$ and the Markov matrix $M^{(N)}$ can be diagonalized by using the Bethe Ansatz [1, 5, 6, 29].

The number of each species of particles is invariant under the action of the transfer matrix $T(\lambda)$ as well as the Markov matrix $(T(\lambda)V_{\mathfrak{s}} \subseteq V_{\mathfrak{s}})$ and we denote the restriction on the sector $\mathfrak{s}$ by $T_{\mathfrak{s}}(\lambda)$. We conjecture that the transfer matrix also satisfies the conjugation relations with the identification matrix and with the new conjugation matrix up to order λ^L: for sectors $\mathfrak{s} = \{s_1 < \cdots < s_N\}$ and $\mathfrak{t} = \mathfrak{s} \setminus \{s_n\}$

$$\varphi_{\mathfrak{t}\mathfrak{s}}T_{\mathfrak{s}}(\lambda) - T_{\mathfrak{t}}(\lambda)\varphi_{\mathfrak{t}\mathfrak{s}} = q^{s_n}\lambda^L\varphi_{\mathfrak{t}\mathfrak{s}}, \tag{63}$$

$$T_{\mathfrak{s}}(\lambda)\psi_{\mathfrak{s}\mathfrak{t}} - \psi_{\mathfrak{s}\mathfrak{t}}T_{\mathfrak{t}}(\lambda) = q^{s_n}\lambda^L\psi_{\mathfrak{s}\mathfrak{t}}. \tag{64}$$

Remark: In the nested algebraic Bethe Ansatz technique, eigenvectors of $M^{(N)}$ are constructed by the action of a product of "*B*-operators" (elements of $\mathcal{T}(\lambda)$) on the vector $|1\cdots1\rangle$. On the other hand, the generalized matrix Ansatz also enables us to construct an eigenvector by the action of a product of ψ's on an *eigenvector* with the same eigenvalue *in a lower sector*. In particular, the stationary state can be written as $\psi\cdots\psi|1\cdots1\rangle$. It seems that our conjugation operator and the *B*-operator play similar roles. However the generalized matrix Ansatz never gives us information about eigenvectors with new eigenvalues in each sector.

5. Conclusion

We have applied the generalized matrix Ansatz to the multispecies ASEP, where the central issue is reduced to finding a representation for the hat algebra. The family of representations that we find here is defined by a recursion formula with respect to the number of species N. This new solution is not obtained by perturbation of the known solution for the TASEP. The SSEP is also a special case where a one-dimensional representation exists. We conjecture that the conjugation relation continues to hold (modulo λ^L) for the Perk-Schultz transfer matrix.

As we remarked in the last section, our generalized matrix Ansatz does not enable us to construct eigenvectors with new eigenvalues in each sector, which is a problem to be solved. Another open question is whether there exists another family of representations to the hat algebra. (In fact we have found different families of representations for the TASEP and SSEP cases.) Applying the technique to open boundary conditions with injection and extraction of particles or other driven-diffusive processes with rules more general than equation (1) can also be an interesting study [7, 8].

[1] F. C. Alcaraz and R. Z. Bariev, 2000, Exact solution of asymmetric diffusion with N classes of particles of arbitrary size and hierarchical order, *Braz. J. Phys.* **30**, 655.
[2] F. C. Alcaraz and V. Rittenberg, 1993, Reaction-diffusion processes as physical realizations of Hecke algebras *Phys. Lett. B* **314**, 377.
[3] C. Arita, 2012, Remarks on the multi-species exclusion process with reflective boundaries, *J. Phys. A* **45**, 155001.
[4] C. Arita, A. Ayyer, K. Mallick and S. Prolhac, 2011, Recursive structures in the multispecies TASEP, *J. Phys. A* **44**, 335004.
[5] C. Arita, A. Kuniba, K. Sakai and T. Sawabe, 2009, Spectrum of a multi-species asymmetric simple exclusion process on a ring, *J. Phys. A* **42**, 345002.
[6] R. J. Baxter, 2007, *Exactly solved models in statistical mechanics*, Dover.
[7] R. A. Blythe and M. R. Evans, 2007, Nonequilibrium steady states of matrix product form: A solver's guide, *J. Phys. A* **40**, R333.
[8] L. Cantini, 2008, Algebraic Bethe Ansatz for the two-species ASEP with different hopping rates, *J. Phys. A: Math. Theor.* **41**, 095001.
[9] V. Chari and A. Pressley, 1995, *A guide to Quantum Groups,* (Cambridge University Press, Cambridge).

[10] B. Derrida, 1998, An exactly soluble non-equilibrium system: the asymmetric simple exclusion process, *Phys. Rep.* **301**, 65.

[11] B. Derrida, 2006, *Matrix ansatz and large deviations of the density in exclusion processes,* Proceedings of the International Congress of Mathematicians Madrid Spain 2006 367-382.

[12] B. Derrida, 2011, Microscopic versus macroscopic approaches to non-equilibrium systems, Proceedings of Statphys 24, *J. Stat. Mech.: Theor. Exp.* P01030.

[13] B. Derrida, M. R. Evans, V. Hakim and V. Pasquier, 1993, An exact solution of a 1D asymmetric exclusion model using a matrix formulation *J. Phys. A* **26**, 1493.

[14] M. R. Evans, P. A. Ferrari and K. Mallick, 2009, Matrix Representation of the Stationary Measure for the Multispecies TASEP, *J. Stat. Phys.* **135**, 217.

[15] P. A. Ferrari and J. B. Martin, 2007, Stationary distributions of multi-type totally asymmetric exclusion processes, *Ann. Prob.* **35**, 807.

[16] O. Golinelli, K. Mallick, 2006, The asymmetric simple exclusion process : an integrable model for non-equilibrium statistical mechanics, *J. Phys. A: Math. Gen.* **39**, 12679.

[17] H. Hinrichsen, S. Sandow and I. Peschel,1996, On matrix product ground states for reaction-diffusion models, *J. Phys. A: Math. Gen.* **29**, 2643.

[18] T. M. Liggett, 1985, *Interacting Particle Systems*, (New-York: Springer).

[19] T. M. Liggett, 1999, *Stochastic Models of Interacting Systems:Contact, Voter and Exclusion Processes*, (New-York: Springer).

[20] C. Kipnis, S. Olla, and S. R. S. Varadhan, 1989, Hydrodynamics and large deviation for simple exclusion processes, *Comm. Pure Appl. Math.* **42**, 115.

[21] P. L. Krapivsky, S. Redner and E. Ben-Naim, 2010, *A Kinetic View of Statistical Physics* (Cambridge: Cambridge University Press).

[22] A. J. Macfarlane, 1989, On q-analogues of the quantum harmonic oscillator and the quantum group $SU(2)_q$ *J. Phys. A:Math. Gen* **22**, 4581.

[23] K. Mallick, 2011, Some exact results for the exclusion process, Proceedings of Statphys 24, *J. Stat. Mech.: Theor. Exp.* P01024.

[24] S. Prolhac, M. R. Evans, K. Mallick, 2009, The matrix product solution of the multispecies partially asymmetric exclusion process, *J. Phys. A* **42**, 165004.

[25] J. H. H. Perk and C. L. Schultz, 1981, New families of commuting transfer matrices in q-state vertex models, *Phys. Lett.* **84A**, 407.

[26] N. Rajewsky, L. Santen, A. Schadschneider and M. Schreckenberg, 1998, The asymmetric exclusion process: Comparison of update procedures, *J. Stat. Phys.* **92** 151.

[27] G. Sandow and G. M. Schütz, 1994, On $U_q[SU(2)]$-Symmetric Driven Diffusion, *Europhysics. Lett.* **26**, 7

[28] B. Schmittmann and R. K. P. Zia, 1995, *Statistical mechanics of driven diffusive systems*, in *Phase Transitions and Critical Phenomena vol 17.*, C. Domb and J. L. Lebowitz Ed., (San Diego, Academic Press).

[29] C. L. Schultz, 1983, Eigenvectors of the Multi-Component Generalization of the six-vertex model, *Physica* **122A**, 71.

[30] G. M. Schütz, 1997, Duality relations for asymmetric exclusion processes, *J. Stat. Phys.* **86**, 1265.

[31] G. M. Schütz, 2001, *Exactly Solvable Models for Many-Body Systems Far from Equilibrium* in *Phase Transitions and Critical Phenomena vol 19.*, C. Domb and J. L. Lebowitz Ed., (Academic Press, San Diego).

[32] E. R. Speer, 1993, *The two species totally asymmetric exclusion process*, in Micro, Meso and Macroscopic approaches in Physics, M. Fannes C. Maes and A. Verbeure Ed. NATO Workshop 'On three levels', Leuven, July 1993.

[33] F. Spitzer, 1970, Interaction of Markov Processes, *Adv. in Math.* **5**, 246.

[34] H. Spohn, 1991, *Large scale dynamics of interacting particles*, (Springer-Verlag, New-York).

[35] S. R. S. Varadhan, 1996, *The complex story of simple exclusion*, in Itô stochastic calculus and probability theory, 385 (Springer, Tokyo).

[36] B. Wehefritz-Kaufmann, 2010, *Dynamical critical exponent for Two-Species Totally Asymmetric Diffusion on a Ring*, Proceedings of the XVIIIth International Colloquium on Integrable Systems and Quantum Symmetries *SIGMA* **6**, 039